DRAGUTIN LJ. DEBELJKOVIĆ[1]
MIĆA B. JOVANOVIC[2]
VESNA DRAKULIĆ[1]

[1]**Faculty of Mechanical Engineering, Belgrade**

[2]**Faculty of Technology and Metallurgy, Belgrade, Yugoslavia**

REVIEW PAPER

66.012-52+66.02/.09:519.876.5

SINGULAR SYSTEM THEORY IN CHEMICAL ENGINEERING THEORY – STABILITY IN THE SENSE OF LYAPUNOV: A SURVEY

Singular systems are those the dynamics of which are governed by a mixture of algebraic and differential equations. These systems also appear in chemical engineering so some mathematical models were been shown to document this fact. The complex nature of singular systems causes many difficultes in the analytical and numerical treatment of such systems, particularly when there is a need for their control. In that sense the question of their stability deserves great attention. A brief survey of the results concerning the stability of a particular class of these systems, operating in free as well in the forced regimes, in the sense of Lyapunov, are presented as the basis for their high quality dynamical investigation.

Singular systems are those the dynamics of which are governed by a mixture of algebraic and differential equations. In that sense the algebrac equations represent the constraints to the solution of the differential part.

These systems are also known as descriptor, semi–state and generalized systems arise naturally as a linear approximation of systems models, or linear system models in many applications such as electrical networks, aircraft dynamics, neutral delay systems, chemical, thermal and diffusion processes, large–scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc. Although singular systems are mostly present in electric and electro–magnetic circuits, in the sequel, will be shown their application in chemical and process technology.

SINGULAR SYSTEMS IN CHEMICAL ENGINEERING

Recently, Bogdanović (1992) has shown that the final superheater stage of a steam generator may have the following mathematical description:

$$\dot{\mathbf{x}}(t)=\mathbf{f}_1\left(\mathbf{x}_1(t),\mathbf{x}_2(t),\mathbf{u}(t),\mathbf{z}(t)\right),$$

$$\mathbf{0}=\mathbf{f}_2\left(\mathbf{x}_1(t),\mathbf{x}_2(t),\mathbf{u}(t),\mathbf{z}(t)\right),$$

$$\mathbf{y}(t)=\mathbf{f}_3\left(\mathbf{x}_1(t),\mathbf{x}_2(t)\right),$$

which is, exactly, one of the possible non–linear representations of singular systems.

It was also, shown that after linearization the mathematical model is of the form:

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}\dot{\mathbf{x}}=$$

$$=\begin{bmatrix} a_{11}&a_{12}&0&0&a_{15}&0&0&a_{18}\\ 0&0&0&0&0&0&0&a_{28}\\ 0&0&0&0&a_{35}&a_{36}&0&0\\ 0&0&a_{43}&0&a_{45}&0&a_{47}&0\\ 0&a_{52}&0&a_{54}&0&0&0&a_{58}\\ a_{61}&0&0&a_{64}&0&0&a_{67}&0\\ 0&a_{72}&0&a_{74}&0&0&a_{77}&0\\ 0&0&0&a_{84}&0&0&a_{87}&a_{88} \end{bmatrix}\mathbf{x}+$$

$$+\begin{bmatrix} b_1\\ b_2\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}\mathbf{u}+\begin{bmatrix} 0\\ 0\\ 0\\ 0\\ f_5\\ 0\\ 0\\ 0 \end{bmatrix}\mathbf{z},$$

$$y=\begin{bmatrix}0&0&0&0&0&0&1&0\end{bmatrix}\mathbf{x},$$

where:

$x_1(t)=i_{ppy}(t)$ – steam enthalphy at the superheater outlet,

$x_2(t)=\rho_{ppy}(t)$ – steam density at the superheater outlet,

$x_3(t)=\theta_{zpp}(t)$ – superheater wall temperature,

$x_4(t)=p_{zpp}(t)$ – steam pressure at the outlet of the superheater,

$x_5(t)=Q_{ppz}(t)$ – heat transfer rate from wall to the superheater,

$x_6(t)=Q_{gypp}(t)$ – heat transfer rate from the gas to the wall,

$x_7(t)=\theta_{ppy}(t)$ – steam temperature at the outlet of the superheater,

Author address: D. Debeljković, Faculty of Mechanical Engineering, Dept. of Control Engineering, University of Belgrade, 27 Marta 80, Belgrade
Paper received: April 10, 2001.
Paper accepted: June 17, 2001.

$x_8(t) = G_{ppy}(t)$ – mass flow rate at the outlet of the superheater,

$u(t) = G_{ppu}(t)$ – mass flow rate at the input of the superheater,

$u(t) = p_{ppu}(t)$ – steam pressure at the input of the superheater,

$y(t) = \theta_{ppi}(t)$ – steam temperature at the outlet of the superheater,

and which is, certainly, a normal canonical description of a singular system.

Another, quite good example of a singular system as a limiting case of a singular by a perturbed process, has been presented in Lapidus et al. (1961).

Namely, the mathematical description of an absorption column may be, basically, given in the following form:

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t) + B_1\mathbf{u}(t),$$

$$\varepsilon\dot{\mathbf{x}}_2(t) = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t) + B_2\mathbf{u}(t),$$

with the initial conditions:

$$\mathbf{x}_1(0) = \mathbf{x}_{10}, \quad \mathbf{x}_2(0) = \mathbf{x}_{20},$$

and the output equation as:

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t),$$

where is a small positive parameter. When $\varepsilon \to 0$ the before mentioned system, obviously, becomes singular. Matrices in this model have the following structure.

$$A_1 = A_4 = \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_1 & a_2 & 0 & 0 & 0 \\ 0 & a_3 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & a_3 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_3 & a_1 \end{bmatrix}$$

where:

$A_2 : \forall a_{ij} \equiv 0$, except $a_{61} = a_2$.

$A_3 : \forall a_{ij} \equiv 0$, except $a_{16} = a_3$.

$a_1 = -1.73$, $a_2 = 0.63$, $a_3 = 0.54$.

$$[B_1 \;\; \varepsilon B_2] = \begin{bmatrix} b_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; [\varepsilon B_3 \;\; B_4] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$b_1 = 0.54$, $b_2 = 0.88$, $\varepsilon = 0.50$.

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and are calculated with real data from the process.

Gas turbines are widely used in a variety of power generation and propulsion applications. The drive for increased efficiency, work ratio and economy is leading to increasingly complex systems with a resulting high demand on the performance of the control system. The turbine dynamics are often complex and vary with operating and ambient conditions. As a result there has been much recent research into the use of robust and adaptive controllers for gas turbines.

In this example a mathematical model of a three shaft gas turbine is considered, Wang, Daley (1993). For the model used, it was shown that a recently proposed adaptive control scheme for singular systems can be applied.

The model used for the simulation study was developed by Foss (1980) and was obtained through the linearization of a thermodynamic model of a typical three–shaft turbofan with reheating. The original nonlinear model was developed using physical laws and the parameters of the linearized model were determined about several operating points. The linearized model takes the following normal state space form:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{x}(t),$$

where (see notation):

$$\mathbf{x}^T \stackrel{\Delta}{=} [n_L, n_1, n_H, p_{2LM}, p_{2L}, p_2, \theta_3, p_{4H}, p_{4I}, p_{4M}, G_H, G_C, p_5, \theta_6],$$

$$\mathbf{u}^T \stackrel{\Delta}{=} [G_{FE}, G_{FR}, A_J],$$

$$\mathbf{y}^T \stackrel{\Delta}{=} [G_1, G_2, \theta_H].$$

n_L – Low pressure shaft speed
n_1 – Intermediate pressure shaft speed
n_H – High pressure shaft speed
p_{2LM} – LP/IP intercompressor pressure
p_{2L} – IP/P intercompressor pressure
p_2 – Combustor pressure
θ_3 – Combustor outlet temperature
p_{4H} – HP/IP interturbine pressure
p_{4I} – IP/LP interturbine pressure
p_{4M} – Post–turbine pressure
G_H – Hot stream mass flow
G_C – Cold stream mass flow
p_5 – Jet pipe pressure
θ_6 – Jet pipe outlet temperature
G_{FE} – Engine fuel
G_{FR} – Reheat fuel
A_J – Nozzle area
G_1 – Fan mass flow
G_2 – HP compressor mass flow

p_6 – Nozzle pressure
θ_{2LM} – LP/IP intercompressor temperature
θ_{2L} – IP/HP intercompressor temperature
θ_H – Thrust.

The system, although fourteenth order is characterised by three dominant eigenvalues and as shown in Daley, Wang (1993), can be represented by a reduced order description:

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}(t) + B_1\mathbf{u}(t)$$

$$\mathbf{x}_2(t) = -B_2\mathbf{u}(t)$$

where:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix}$$

and:

$$\mathbf{x}_1(t) \overset{\Delta}{=} [n_L, n_I, n_H]$$

$$\mathbf{x}_2(t) \overset{\Delta}{=} [p_{2LM}, p_{2L}, p_2, \theta_3, p_{4H}, p_{4I}, p_{4M}, G_H, G_C, p_5, \theta_6].$$

These equqtions can equivalently be expressed by a generalised state–space description:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{x}(t),$$

where E is the singular matrix

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

It is a convenient structure that can be used to start dynamical analysis, synthesis or to develop an adaptive controller, Daley, Wang (1993).

POSSIBILITIES OF DYNAMICAL ANALYSIS OF SINGULAR SYSTEMS: SYSTEMS STABILITY FEATURES IN THE SENSE OF LYAPUNOV: A SURVAY

Consider linear singular systems (LSS) represented, by:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t),\ \mathbf{x}(t_o) = \mathbf{x}_o, \quad \mathbf{y}(t) = C\mathbf{x}(t), \tag{1}$$

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),\ \mathbf{x}(t_o) = \mathbf{x}_o, \quad \mathbf{y}(t) = C\mathbf{x}(t), \tag{2}$$

with the matrix E possibly singular, where $\mathbf{x}(t) \in \mathbf{R}^n$ is a generalized state–space vector and $\mathbf{u}(t) \in \mathbf{R}^m$ is a control variable. Matrices A, B and C are of the appropriate dimensions and are defined over the field of real numbers. System given by eq. (1) is operatinig in a free and system given by eq. (2) is operating in a forced regime, i.e. some external force is applied on it. It should be stressed that, in the general case, the initial conditions for an autonomus and a system operating in the forced regime need not be the same. System models of this form have some important advantages in comparison with models in the normal form, e.g. when E = I and an appropriate discussion can be found in Bajic (1992) and Debeljkovic et al. (1996, 1996a, 1998).

The complex nature of singular systems causes many difficultes in analytical and numerical treatment that do not appear when systems in the normal form are considered. In this sense questions of existence, solvability, uniqueness, and smothness are present which must be solved in satisfactory manner. A short and concise, acceptable and understandable explanation of all these questions may be found in the papers of Debeljkovic (2001) and Lazarevic et al. (2001).

The survey of updated results for singular systems and a broad bibliography can be found in Bajic(1992), Campbell (1980, 1982), Lewis (1986, 1987), Debeljkovic et al. (1996.a, 1996.b, 1998) and in the two special issues of the journal Circuits, Systems and Signal Procesing (1986, 1989).

STABILITY IN THE SENSE OF LYAPUNOV

Stability plays a central role in the theory of systems and control engineering.There are different kinds of stability problems that arise in the study of dynamic systems, such as Lyapunov stability, finite time stabilty, practical stability, technical stabilty and BIBO stability. The first part of this section is concerned with the stability of the equlibrium points in the sense of Lyapunov stability of linear autonomous singular systems. As we treat the linear systems this is equivalent to the study of the stability of the systems. The Lyapunov direct method is well exposed in a number of very well known references. Here we present some different and interesting approaches to this problem, including the contributions of the authors of this paper.

LINEAR AUTONOMOUS SINGULAR SYSTEMS

Stability definitions

Definition 1. Eq.(1) is exponentially stable if one can find two positive constants α, β such that $\|\mathbf{x}(t)\| \le \beta\|\mathbf{x}_o\|e^{-\alpha t}$ for every solution of Eq.(1), Pandolfi (1980).

Definition 2. The system given by Eq.(1) will be termed asymptotically stable iff, for all consistent initial conditions $\mathbf{x}_o$, $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$, Owens, Debeljković (1985).

Definition 3. We call system given by Eq. (1) asymptotically stable if all roots of det $(sE - A)$, i.e. all finite eigenvalues of this matrix pencil, are in the open left – half complex plane, and system under consideration is impulsive free if there is no $\mathbf{x}_o$ such that $\mathbf{x}(t)$ exibits discontinuous behavior in the free regime, Lewis (1986).

Definition 4. The system given by Eq. (1) is called asymptotically stable iff all finite eigenvalues λ_i, $i = 1,..., n_1$, of the matrix pencil $(\lambda E - A)$ have negative parts, Muller (1993).

Definition 5. The equilibrium $x = \mathbf{0}$ of system given by Eq.(1) is said to be stable if for every $\varepsilon > 0$, and for any $t_o \in J$, there exists a $\delta = \delta(\varepsilon, t_o) > 0$, such that $\| x(t, t_o, \mathbf{x}_o)\| < \varepsilon$ hold for all $t \geq t_o$, whenever $\mathbf{x}_o \in W_k$ and $\|\mathbf{x}_o\| < \delta$, where J denotes time interval such that $J = [t_o, +\infty)$, $t_o \geq 0$, Chen, Liu (1997).

Definition 6. The equilibrium $x = \mathbf{0}$ of a system given by Eq. (1) is said to be unstable if there exist a $\varepsilon > 0$, and $t_o \in J$, for any $\delta > 0$, such that there exists a $t^* \geq t_o$, for which $\| x(t^*, t_o, \mathbf{x}_o)\| \geq \varepsilon$ holds, although $\mathbf{x}_o \in W_k$ and $\|\mathbf{x}_o\| < \delta$, Chen, Liu (1997).

Definition 7. The equilibrium $x = \mathbf{0}$ of a system given by Eq. (1) is said to be attractive if for every $t_o \in J$, there exists an $\eta = \eta(t_o) > 0$, such that $\lim_{t\to\infty} \mathbf{x}(t, t_o, \mathbf{x}_o) = 0$, whenever $\mathbf{x}_o \in W_k$ and $\|\mathbf{x}_o\| < \eta$, Chen, Liu (1997).

Definition 8. The equilibrium $x = \mathbf{0}$ of a singular system given by Eq. (1) is said to be asymptotically stable if it is stable and attractive, Chen, Liu (1997).

Lemma 1. The equilibrium $x = \mathbf{0}$ of a linear singular system given by Eq. (1) is asymptotically stable if and only if it is impulsive–free, and $\sigma(E,A) \subset C^-$ Chen, Liu (1997).

Lemma 2. The equilibrium $x = \mathbf{0}$ of a system given by Eq. (1) is asymptotically stable if and only if it is impulsive–free, and $\lim_{t\to\infty} \mathbf{x}(t) = 0$, Chen, Liu (1997).

Stability theorems

Theorem 1. Eq. (1), with A = I, I being the identity matrix, is exponentially stable if and only if the eigenvalues of E have nonpositive real parts.

Proof. The state response of singular system, under consideration, is given by:

$$\mathbf{x}(t) = e^{-E^D A(t-t_o)}\, EE^D\, \mathbf{q},\ \mathbf{q} \in C^n \tag{3}$$

with the restriction on the vector of consistent initial conditions, given by the following equation:

$$\mathbf{x}_o = EE^D \mathbf{x}_o\,. \tag{4}$$

If E is written in diagonal form, then:

$$e^{-E^D A(t-t_o)} EE^D = \begin{bmatrix} e^{Q_o^{-1}t} & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

which decays exponentially when $\lambda \in \sigma(0)$ implying that $Re(\lambda) < 0$, where $\sigma(\lambda_i)$ denotes the eigenvalue spectar of the appropriate matrix. We use upper index "D" to indicate the Drazin inverz.

Because the eigenvalues of Q_o are those eigenvalues of E which are not zero, it has completed the proof.

Theorem 2. Let I_Ω be the matrix which represents the oprator on $\mathbf{R}^n$ which is the identity on Ω and the zero operator on Λ. Eq. (1), with A = I, is stable if an n x n matrix P exist, which is the solution of the matrix equation:

$$E^T P + PE = -I_\Omega, \tag{6}$$

with the following propreties:

i) $P = P^T$,
ii) $P\mathbf{q} = \mathbf{0}$, $\mathbf{q} \in \Lambda$
iii) $\mathbf{q}^T P\mathbf{q} > 0$, $\mathbf{q} \neq \mathbf{0}$, $\mathbf{q} \in \Omega$,

where:

$$\Omega = W_k = \aleph\,(I - EE^D) \tag{7}$$

$$\Lambda = \aleph\,(EE^D) \tag{8}$$

where W_k is the subspace of consistent intial conditions. $\aleph$ denotes the kerrnel or null space of the matrix ( ).

Proof. If Eq. (6) has a solution P as above, E cannot have eigenvalues with positive real parts. Hence, Eq. (1) is stable. Conversely, assume that Eq. (1) is stable. Let P be defined by:

$$\mathbf{q}^T P\mathbf{q} = \int_0^{+\infty} \|\exp(Et)E^D\mathbf{q}\|^2\, dt\,. \tag{9}$$

The integral is zero if $\mathbf{q} \in \Lambda$ and is a finite number if $\mathbf{q} \in \Lambda$. It is clear that matrix P is solution of Eq. (6) with the properties, a), b), c), Pandolfi (1980).

Theorem 3. The system given by Eq. (1) is asymptotically stable if and only if:

a) A is invertible and

b) a positive–definite, self–adjoint operator P on $\mathbf{R}^n$ exist, such that:

$$\text{c) } A^T PE + E^T PA = -Q \tag{10}$$

where Q is self–adjoint and positive in the sense that:

$$\mathbf{x}^T(t)\, Q\mathbf{x}(t) > 0 \text{ for all } \mathbf{x} \in W_{k^*} / \{\mathbf{0}\}, \tag{11}$$

Owens, Debeljković (1985).

Proof. To prove sufficiency, note that $W_k \cap \aleph(E) = \{0\}$ indicates that:

$$V(x) = \mathbf{x}^T(t)E^T PE\mathbf{x}(t) \tag{12}$$

is a positive–definite quadratic form on W_{k^*}. All smooth solutions $\mathbf{x}(t)$ evolve in W_{k^*} so V(x) can be used as a "Lyapunov function". Clearly, using the equation of motion Eq. (1), one can have:

$$\begin{aligned}
\dot{V} &= \dot{\mathbf{x}}^T(t)\, E^T\, PE\, \mathbf{x}(t) + \mathbf{x}^T(t)\, E^T\, PE\, \dot{\mathbf{x}}(t) = \\
&= (E\mathbf{x}(t))^T\, PE\, \mathbf{x}(t) + \mathbf{x}^T(t)\, E^T\, PE\, \dot{\mathbf{x}}(t) = \\
&= (A\mathbf{x}(t))^T\, E^T\, PE\, \mathbf{x}(t) + \mathbf{x}^T(t)\, E^T\, PA\, \mathbf{x}(t) = \\
&= \mathbf{x}(t)\, A^T\, PE\, \mathbf{x}(t) + \mathbf{x}^T(t)\, E^T\, PA\, \mathbf{x}(t) = \\
&= -\mathbf{x}^T(t)\, Q\, \mathbf{x}(t) \leq -\lambda V\,,
\end{aligned} \tag{13}$$

where:

$$\lambda = \min\, \{\, \mathbf{x}^T(t)Q\mathbf{x} : V(x) = 1,\ \mathbf{x} \in W_{k^*}\}, \tag{14}$$

is strictly positive by Eq. (11).

Clearly:

$$0 \leq V(\mathbf{x}(t)) \leq V(\mathbf{x}_o)e^{\lambda t} \to 0\ (t \to \infty), \tag{15}$$

so that $E\mathbf{x}(t)$ and $\mathbf{x}(t)$ tend to zero as $t \to \infty$ as required, Debeljković et al. (1996a).

Theorem 4. The system given by Eq. (1) is asymptotically stable if and only if:

a) a is invertible and

b) a positive–definite, self–adjoint operator P exist, such that:

$$\mathbf{x}(t)\ (A^TPE + E^TPA)\mathbf{x}(t) = -\mathbf{x}^T(t)I\mathbf{x}(t) \text{ for all } \mathbf{x}\in W_{k^*} \tag{16}$$

Owens, Debeljković (1985).

Theorem 5. Let (E, A) be regular and (E, A, C) be observable. Then (E, A) is impulsive free and assymptotically stable if and only if a positive definite solution P to:

$$A^TPE + E^TPA + E^TC^TCE = 0, \tag{17}$$

exist and if P_1 and P_2 are two such solutions, then $E^TP_1E = E^TP_2E$, Lewis (1986).

Theorem 6. If there are symetric matrices P, Q satisfying:

$$A^TPE + E^TPA = -Q \tag{18}$$

and if:

$$\mathbf{x}E^TPE\mathbf{x} > 0 \quad \forall \mathbf{x} = S_1\mathbf{y}_1 \neq \mathbf{0}, \tag{19}$$

$$\mathbf{x}^TQ\mathbf{x} \geq \quad \forall \mathbf{x} = S_1\mathbf{y}_1, \tag{20}$$

then the system described by Eq. (1) is asymptotically stable if:

$$\operatorname{rank}\begin{bmatrix} sE-A \\ S_1^T Q \end{bmatrix} = n \quad \forall s\in \mathbf{C}, \tag{21}$$

and marginally stable if the condition given by Eq. (21) does not hold, Muller (1993).

Proof. Assume P, Q according to Eq. (19, 20), then by transformation:

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad S = [S_1 \quad S_2] \tag{22}$$

$$RES = \begin{bmatrix} I_1 & 0 \\ 0 & N_v \end{bmatrix}, \quad RAS = \begin{bmatrix} A_1 & 0 \\ 0 & I_2 \end{bmatrix} \tag{23}$$

where the identity matrices I_1 and I_2 are of dimension n_1 and n_2 with $n_1 + n_2 = n$ and the n_2 x n_2 matrix N_v is the nilpotent of index v, one has:

$$A_1^T P_1 + P_1A_1 = -S_1^TQS_1 = -Q_1, \tag{24}$$

with:

$$P_1 = P_1^T > 0, \quad Q_1 = Q_1^T \geq 0, \tag{25}$$

Therefore the system given by Eq. (1) is stable in the sense of Lyapunov and is assymptotically stable if and only if:

$$\operatorname{rank}\begin{bmatrix} sI_1-A_1 \\ Q_1 \end{bmatrix} = n_1, \quad \forall s\in \mathbf{C}, \tag{26}$$

So, it is necessary to show that the condition:

$$\operatorname{rank}\begin{bmatrix} sE-A \\ S_1^TQ \end{bmatrix} = n, \quad \forall s\in \mathbf{C}, \tag{27}$$

is equivalent to the expression, given by Eq. (26). By the transformation of Eqs. (22–23) one has:

$$\operatorname{rank}\begin{bmatrix} sE-A \\ S_1^TQ \end{bmatrix} = \operatorname{rank}\begin{bmatrix} sI_1-A_1 & 0 \\ 0 & sN_v-I_2 \\ Q_1 & Q_{12} \end{bmatrix} \tag{28}$$

showing the equivalence of Eq. (26) and Eq. (27).

Theorem 7. The equilibrium $\mathbf{x} = \mathbf{0}$ of a system given by Eq. (1) is asymptotically stable, if an n x n symmetric positive definite matrix P exist, such that along the solutions of system, given by Eq. (1), the derivative of function $V(E\mathbf{x}) = (E\mathbf{x})^T P(E\mathbf{x})$, is a negative definite for the variates of $E\mathbf{x}$, Chen, Liu (1997).

Proof. First, the regularity of (E, A) means that n x n nonsingular matrices U and V exist, such that:

$$UEV = \begin{pmatrix} I_1 & 0 \\ 0 & N \end{pmatrix}, \quad UAV = \begin{pmatrix} A_1 & 0 \\ 0 & I_2 \end{pmatrix}, \tag{29}$$

and Eq. (1) is equivalent to:

$$\begin{aligned} \dot{\mathbf{z}}_1 &= A_1\mathbf{z}_1 + \mathbf{0} \\ N\dot{\mathbf{z}}_2 &= \mathbf{0} + \mathbf{z}_2 \end{aligned} \tag{30}$$

here $Q(\mathbf{Z}_1 \ \ \mathbf{X}_2)^T = \mathbf{x}$, a_1 is an n_1 x n_1 nonsingular matrix and N is an n_2 x n_2 nilpotent matrix, $n_1 + n_2 = n$.

Next, the fact that $V(E\mathbf{x})$ is a negative definite quadratic form for the variates of $E\mathbf{x}$ means that an n x n symetric matrix W exists with E^TWE is a positive semi definite with the rank of E^TWE being equal to r, such that:

$$V(E\mathbf{x}) = -(E\mathbf{x})^T W(E\mathbf{x}) \tag{31}$$

or:

$$A^TPE + E^TPA = -E^TWE. \tag{32}$$

Letting:

$$P = U^T\begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}U, \tag{33}$$

$$W = U^T\begin{pmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{pmatrix}U, \tag{34}$$

one has:

$$\begin{aligned} &P_{11}A_1 + A_1^TP_{12} = -W_{11} \\ &P_{22}N + N^TP_{22} = -N^TW_{22}N, \\ &P_{12} + A_1^TP_{12}N = -W_{12}N \end{aligned} \tag{35}$$

where P_{11}, P_{22} and are all positive definite matrices.

In the following it proven that N = 0. Suppose that the form of nilpotent matrix N is

$$N = \begin{pmatrix} J_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & J_i \\ & & & & & 0 \end{pmatrix}, \tag{36}$$

where J_i is a Jordan block matrix in which the diagonal elements are all zero (i = 1,..., s), then all elements of the first row of both N^TP_{22} and N^TW_{22} N are zero. It follows from the second formula of Eq. (35) that all elements of first row $P_{22}N$ are zero. If N = 0 is not true, without loss of generality, this suposes that $J_1 \neq 0$, then it can be deduced that the element of the first row and first column of matrix P_{22} is zero. This is not true since P_{22} is positive definite.

Thus must be N = 0, in other words, and the linear singular system described by Eq. (1) is impulse–free.

The positive definitity of matrix and the first formula of Eq. (35) imply that is an asymptotically stable matrix. It follows from Eq. (30) and N = 0 that $\lim_{t\to\infty} \dot{\mathbf{x}} = \mathbf{0}$ hold from $\mathbf{x}$ = Q ($\mathbf{z}_1$ $\mathbf{z}$)^T and the conclusion of Theorem 7 follows directly from Lemma 1.

Theorem 8. If an n x n symmetric, positive definite matrix P exists, such that along with the solutions of system, given by Eq. (1), the derivative of the function $V(E\mathbf{x}) = (E\mathbf{x})^T P(E\mathbf{x})$ i.e. $\dot{V}(E\mathbf{x})$ is a positive definite for all variates of E**x**, then the equilibrium $\mathbf{x} = \mathbf{0}$ of the system given by Eq. (1) is unstable, Chen, Liu (1997).

Theorem 9. If an nxn symmetric, positive definite matrix P exists, such that along with the solutions of system, given by Eq. (1), the derivative of the function $V(E\mathbf{x}) = (E\mathbf{x})^T P(E\mathbf{x})$ i.e. $\dot{V}(E\mathbf{x})$ is negative semidefinite for all variates of E**x**, then the equilibrium $\mathbf{x} = \mathbf{0}$ of the system, given by Eq. (1), is stable, Chen, Liu (1997).

Theorem 10. Let (E, A) be regular and (E, A, C) be impulse observable and finite dynamics detectable. Then (E, A) is stable and impulse–free if and only if a solution (P, H) to the generalized Lyapunov equations (GLE) exists.

$$A^TP + H^TA + C^TC = 0, \tag{37}$$

$$H^TE = E^TP \geq 0 \tag{38}$$

Proof. We assume that E, A, C are given by a Weierstrass form

$$E = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix},$$
$$C = [C_1 \;\; C_2], \tag{39}$$

where r is the number of finite dynamic modes, and N is a nilpotent Jordan form.

Sufficiency. Partitioning:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{40}$$

one obtains:

$$H_{11}^TA_1 + A_1^TP_{11} + C_1^TC_1 = 0,$$
$$H_{11}^T = P_{11} \geq 0, \tag{41}$$

$$H_{12}^TA_1 + P_{21} + C_2^TC_1 = 0,$$
$$H_{12}^T = N^TP_{21}, \tag{42}$$

$$H_{21}^T + A_1^TP_{12} + C_1^TC_2 = 0,$$
$$H_{21}^TN = P_{12}, \tag{43}$$

$$H_{22}^T + P_{22} + C_2^TC_2 = 0,$$
$$H_{22}^TN = N^TP_{22}, \tag{44}$$

Note that (E, A, C) is impulse observable if and only if:

$$\Re(N^T) + \Re(C_2^T) + \aleph(N^T) = \mathbf{R}^{n-r}. \tag{45}$$

Let

$$\alpha: = \min\{k \mid (N^T)^k = 0, k > 0\}. \tag{46}$$

Then:

$$\Re(N^T)^{\alpha-1} = \Re(N^T)^{\alpha-1} C_2^T + \Re(N^T)^{\alpha} =$$
$$= \Re(N^T)^{\alpha-1}C_2^T \tag{47}$$

Pre–mutiplying Eq. (45) by $(N^T)^{\alpha-1}$ and post–multiplying by $(N)^{\alpha-1}$ yields:

$$(N^T)^{\alpha-1} H_{22}^T (N^{\alpha-1} + (N^T)^{\alpha-1} P_{22}(N)^{\alpha-1} =$$
$$= -(N^T)^{\alpha-1} C_2^T C_2 (N)^{\alpha-1}. \tag{48}$$

It follows again from Eq. (45) that both terms in the left–hand side of Eq. (48) are zero, so that $(N^T)^{\alpha-1} C_2^T = 0$. Hence, from Eq. (48), one obtains $\Re(N^T)\alpha^{-1} = 0$, contradicting the minimality of α. This implies that N = 0, so that (E, A) is impulse–free. Also, since (A_1, C_1) is detectable, one can see from eq. (41) that A_1 is stable. Hence (E, A) is stable, Takaba et al. (1995).

Necessity. Suppose that (E, A) is stable and impulse–free. Then Eqs. (41 – 44) are with N = 0. From the hypotheses, there exists a solution $P_{11} \geq 0$ to Eq. (40). Moreover, $P_{12} = H_{12} = 0$, $P1 = H_{21} = -C_2^TC_1$, and P_{22}, H_{22} are arbitrary satisfying Eq. (45). Thus it has been shown that a solution (P, H) exists to Eqs. (37 – 38) with:

$$E^TP = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \geq 0. \tag{49}$$

Takaba et al. (1995).

Some assumptions and preliminaries are needed for further exposures.

Suposse that matrices E and A commute that is: EA = AE. Then a real number λ exists such that λE–I = A, otherwise, from the property of regularity, one may multiply Eq. (1) by $(\lambda E-A)^{-1}$ so one can obtain the system that satisfy the above assumption and keep the stability the same as the original system.

It is well known that there always exists linear nonsingular transformation, with invertible matrix T, such that:

$$[TET^{-1} \;\; TAT^{-1}] =$$
$$= \{\mathrm{diag}[E_1 \;\; E_2]\; \mathrm{diag}[A_1 \;\; A_2]\} \tag{50}$$

where E_1 is of full rank and E_2 is a nilpoptent matrix, satisfying:

$$E_2^h \neq 0\,, \quad E_2^{h+1} = 0\,, \quad h \geq 0\,. \tag{51}$$

In addition, it is evident:

$$A_1 = \lambda E_1 - I_1, \;\; A_2 = \lambda E_2 - I_2. \tag{52}$$

The system, given by Eq. (1), is equivalent to:

$$E_1\, \mathbf{x}_1(t) = A_1\, \mathbf{x}_1(t) + B_1\, \mathbf{u}(t)\,, \tag{53a}$$

$$E_2\, \dot{\mathbf{x}}_2 = A_2\, \mathbf{x}_2(t) + B_2\, \mathbf{u}(t)\,, \tag{53b}$$

where $\mathbf{x}^T = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}$

Lemma 3. The system, given by Eq. (1), is asymptotically stable if and only if the "slow" sub – system, Eq. (53a) is asymptotically stable, Zhang et al. (1998a)

Lemma 4. $\mathbf{x}_1 \neq \mathbf{0}$ is equivalent to $E^{h+1}\mathbf{x} \neq \mathbf{0}$, Zhang et al. (1998a).

Define Lyapunov function as:

$$V(E^{h+1}\mathbf{x}) = \mathbf{x}^T\,(E^{h+1})^T\,PE^{h+1}\mathbf{x}, \tag{54}$$

where:

$P > 0$, $P \in \mathbf{R}^{nxn}$ satisfying:

$V(E^{h+1}\mathbf{x}) > 0$ if $E^{h+1}\mathbf{x} \neq \mathbf{0}$, when $V(\mathbf{0})=0$.

From Eq. (1) and Eq. (53), bearing in mind that $EA = AE$, one can obtain:

$$(E^h)^T\,A^T\,PE^{h+1} + (E^{h+1})^T\,PAE^h = \\ = -(E^{h+1})^T\,WE^{h+1} \tag{55}$$

where $W > 0$, $W \in \mathbf{R}^{nxn}$.

Eq. (55) is said to be Lyapunov equation for a system given by Eq. (1).

Denote with:

$$\text{degre}\,\det(sE - A) = \text{rank}E_1 = r. \tag{56}$$

Theorem 11. The system, given by Eq. (1), is asymptotically stable if and only if for any matrix $W > 0$, Eq. (55) has a solution $P \geq 0$ with a positive external exponent r, Zhang et al. (1998a).

Proof.

Necessity. Eq. (53) with $\mathbf{u}(t) = \mathbf{0}$ is substituted into Eq. (55), obtains:

$$\begin{bmatrix} (E_1^h)^T & 0 \\ 0 & (E_2^h)^T \end{bmatrix} \begin{bmatrix} A_1^T & 0 \\ 0 & A_2^T \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix} \begin{bmatrix} E_1^{h+1} & 0 \\ 0 & 0 \end{bmatrix} + \\ + \begin{bmatrix} E_1^{h+1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} = \\ = - \begin{bmatrix} (E_1^{h+1})^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix} \begin{bmatrix} E_1^{h+1} & 0 \\ 0 & 0 \end{bmatrix} \tag{57}$$

Notice that E_1 is of full rank, so the equivalent form can be obtained:

$$A_1^T P_1 E_1 + E_1^T P_1 A_1 = -E_1^T W_1 E_1\,, \tag{58a}$$

$$P_2 A_2 E_2^h = 0\,, \tag{58b}$$

where:

$$T^T P T^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, \tag{59}$$

$$T^T W T^{-1} = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix}, \tag{60}$$

If matrix pair (E, A) is asymptotically stable thus, implies that (E_1, A_1) is asymptotically stable, too.

Let $W > 0$, then $W_1 > 0$. Then Eq. (58a) has a solution $P_1 > 0$ with an internal exponent r. Let $P_2 = 0$ then $P_3 = 0$, and the necessity is proved.

Sufficiency. For any $W > 0$ implies $W_1 > 0$, so Eq. (55) has a solution if and only if Eq. (58a) and Eq. (58b) have solutions respectively, and $P_1 > 0$. Therefore (E_1, A_1) is asymptotically stable. Then the sufficency follows immediattly from Lemma 3.

One can choose $P_3 > 0$ since is not restricted and one can have the following result immediately.

Theorem 12. The system, given by Eq. (1), is asymptotically stable if and only if for any given $W > 0$ the Lyapunov Eq. (55) has the solution $P > 0$, Zhang et al (1998a).

The conclusion is the same as in the case of the very well known Lyapunov stability theory if E is of full rank. If matrix E is singular then there is more than one solution.

It should be noted that the results of the preceeding theorems are very similar in some way and are derived only for regular linear singular systems.

In order to investigate the stability of irregular singular systems, the folowing results can be used, Bajić at al. (1992). For this case, the linear singular system is used in the suitable canonical form, i.e.:

$$\dot{\mathbf{x}}_1(t) = A_1\,\mathbf{x}_1(t) + A_2\,\mathbf{x}_2(t)\,, \tag{61}$$

$$\mathbf{0} = A_3\,\mathbf{x}_1(t) + A_4\,\mathbf{x}_2(t)\,. \tag{62}$$

Herewith, we examine the problem of the existence of solutions which converge toward the origin of the systems phase–space for regular and irregular singular linear systems. By a suitable nonsingular transformation, the original system is transformed to a convenient form. This form of system equations enables development and easy application of Lyapunov's diect method (LDM) for the intended existence analysis for a subclass of solutions. In the case when the existence of such solutions is established, an understimation of the weak domain of the attraction of the origin is obtained on the basis of symmetric positive definite solutions of a reduced order matrix Lyapunov equation. The estimated weak domain of attraction consists of points of the phase space, which generate at least one solution convergent to the origin.

Let as, before, the set of the consistent initial values of Eqs. (61 – 62) be denoted by W_{k^*}. Also, consider the manifold $\mathbf{m} \subseteq \mathbf{R}^{nxn}$. determined by Eq. (62) as $\mathbf{m} = \aleph([A_3 \;\; A_4])$. For the system governed by Eq. (61–62) the set W_{k^*} of the consistent initial values is equal to the manifold $\mathbf{m}$, that is $W_{k^*} = \mathbf{m}$.

It is easy to see, that the convergence of the solutions of system given by Eq.(1) and system, given by Eqs. (61 – 62), toward the origin is an equivalent problem, since the proposed transformation is nonsingular.

Thus, for the null solution of Eqs. (61 – 62), the weak domain of attraction is going to be estimated. The weak domain of attraction of the null solution $\mathbf{x}(t) \equiv 0$ of system given by Eq. (61 – 62) is defined by:

$$\mathbf{D} \stackrel{\Delta}{=} \{\mathbf{x}_o \subseteq \mathbf{R}^n : \mathbf{x}_o \in \mathbf{m}, \exists \mathbf{x}(t, \mathbf{x}_o), \lim_{t \to \infty} \| \mathbf{x}(t, \mathbf{x}_o) \| \to \mathbf{0} \} . \tag{63}$$

The term weak is used because solutions of Eqs. (61 – 62) need not to be unique, and thus for every $\mathbf{x}_o \in \mathbf{D}$ there may also exist solutions which do not converge toward the origin. In our case $\mathbf{D} = \mathbf{m} = W_{k^*}$, and the weak domain of attraction may be thought of as the weak global domain of attraction Note that this concept of global domain of attraction used in the paper, differs considerably with respect to the global attraction concept known for state variable systems, Bajić et al. (1992), Debeljković et al. (1996).

Our task is to estimate the set **D**. We will use LDM to obtain the underestimate **D** of the set **D** (i.e. $\mathbf{D_e} \subseteq \mathbf{D}$). Our development will not require the regularity condition of the matrix pencil ($sE - A$).

For the systems in the form of Eqs. (61 – 62) the Lyapunov–like function can be selected as:

$$V(\mathbf{x}(t)) = \mathbf{x}_1^T(t) P \mathbf{x}_1(t), \; P = P^T, \tag{64}$$

where P will be assumed to be a positive definite and real matrix. The total time derivative of V along the solutions of Eqs. (61 – 62) is then:

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}_1^T(t)\,(A_1^T P + PA_1)\,\mathbf{x}_1(t) + \mathbf{x}_1^T(t) PA_2\,\mathbf{x}_2(t) + \mathbf{x}_2^T(t) A_2^T P \mathbf{x}_1(t) . \tag{65}$$

A brief consideration of the attraction problem shows that if Eq. (65) is negative definite, then for every $\mathbf{x}_o \in W_{k^*}$ we have $\|\mathbf{x}_1(t)\| \to 0$ as $t \to \infty$. Then $\|\mathbf{x}_2(t)\| \to 0$ as $t \to \infty$, for all those solutions for which the following connection between $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ holds:

$$\mathbf{x}_2(t) = L\mathbf{x}_1(t), \; \forall t \in \mathbf{R} \tag{66}$$

The main question is if the relation Eq. (66) can be established in a way so as not to contradict the constraints. Since it is not possible for irregular singular linear system, then we have to reformulate our task to establish the relation Eq. (66) so that it does not pose to many addtional novel constraints to Eq. (62).

In order to efficiently use this fact for the analysis of the attraction problem, we introduce the following consideration that also proposes a construction of the matrix L.

Let Eq. (66) hold. Assume that the rank condition:

$$\mathrm{rank}\,[A_3 \;\; A_4] = \mathrm{rank}\, A_4 = r \le n_2, \tag{67}$$

is satisfied. Then a matrix L exist, Tseng and Kokotović (1988), being any solution of the matrix equation:

$$0 = A_3 + A_4 L, \tag{68}$$

where 0 is a null matrix of dimensions the same as A_3.

On the basis of Eq. (66), eq. (68) and Eq. (62), it becomes evident that whenever a solution $\mathbf{x}(t)$ fufills Eq. (66), then it has also has to fulfill Eq. (62). One can investigate in more detail the implications of the introduced equations. When they hold, then all solutions of the system Eqs. (61 – 62), which satisfy Eq. (66), are subject to algebraic constraints:

$$F\mathbf{x}(t) = \begin{bmatrix} A_3 & A_4 \\ L & -I \end{bmatrix} \mathbf{x}(t) = 0 . \tag{69}$$

Assuming that $\dot{V}(\mathbf{x}(t))$ determined by Eq. (65) is a negative definite, the following conclusions are important:

1. The solution of Eqs. (61 – 62) has to belong to set $\aleph([A_3 \;\; A_4]) \cap \aleph([L - I])$;

2. If rank F = n then judgement on the domain of attraction of the null solutionis not possible on the basis of the adopted approach, or more precisely, in this case the estimate of the weak domain **D** of attraction is a singleton: $\{\mathbf{x}(t) \in \aleph([A_3 \;\; A_4]) : \mathbf{x}(t) \equiv 0\}$;

3. If rank F < n, then the estimates of the weak domain of attraction needs to be a singleton and it is estimated as:

$$\mathbf{D_e} = \{\mathbf{x}(t) \in \Re^n : \mathbf{x}(t) \in \aleph([A_3 \;\; A_4]) \cap \aleph([L - I])\} \subseteq \mathbf{D}. \tag{70}$$

Now Eq. (65) and Eq. (66) are employed to obtain:

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}_1^T(t)\,((A_1 + A_2 L)^T P + P(A_1 + A_2 L))\,\mathbf{x}_1(t) , \tag{71}$$

which is a negative definite with respect to $\mathbf{x}_1(t)$ if and only if:

$$\Omega^T P + P\Omega = -Q, \quad \Omega = A_1 + A_2 L, \tag{72}$$

where Q is real a symmetric positive definite matrix. We are now in the position to state the following result.

Theorem 13. Let the rank condition Eq. (67) hold and let rank F < n, where the matrix F is defined in Eq. (69). Then, the underestimate $\mathbf{D_e}$ of the weak domain **D** of the attraction of the null solution of system given by Eqs. (61 – 62), is determinated by Eq. (70), providing ($A_1 + A_2 L$) is a Hurwitz matrix. If $\mathbf{D_e}$ is not a singleton, then there are solutions of Eq. (61–62) different form null solution, $\mathbf{x}(t) \equiv \mathbf{0}$, which converge toward the origin as time $t \to +\infty$.

Proof. If the rank condition is satisfied, then for all solutions of Eqs. (61 – 62) that satisfy Eq. (66), one can have $\mathbf{x}(t) \in \aleph([L - I])$ and simultaniously, these solutions $\mathbf{x}(t) \in \mathbf{m} \equiv \aleph([A_3 \;\; A_4])$. Hence, according to Eq. (69), $\mathbf{x}(t) \in \aleph([A_3 \;\; A_4]) \cap \aleph([L - I])$. However, Eq. (65) and Eq. (66) implies Eq. (71). Since ($A_1 + A_2 L$) is a Hurwitz matrix, then according to the well known results on the

Lyapunov matrix equation, a unique symmetric positive definite matrix P satisfying Eq. (72) exists. Hence, V defined by Eq. (64) is a positive definite function with respect to $\mathbf{x}_1(t)$, and its total time derivative along the solutions of Eqs. (61 – 62) constrained by eq. (66) is a negative definite, so $\lim\|\mathbf{x}_1(t)\| \to 0$ as $t \to +\infty$, as long as $\mathbf{x}_0 \in \aleph([A_3 \;\; A_4]) \cap \aleph([L - I])$. But Eq. (66) implies also $\lim \|\mathbf{x}_2(t)\| = \lim \|\mathbf{x}_1(t)\| \to 0$ as $t \to +\infty$. So, with rank F < n, more than one value of $\mathbf{x}(t)$ satisfies Eq. (69). Hence, as $\aleph([A_3 \;\; A_4]) \cap \aleph([L - I])$ is not a singleton, solutions, different form null solutions exist, which converge toward the origin as time $t \to +\infty$. This proves the theorem, Bajic et al. (1992).

LINEAR NON–AUTONOMOUS SINGULAR SYSTEMS

In the sequel, the generalized Lyapunov equations (GLE) given by Bender (1987) are further studied for continuous–time singular systems. Under a rank condition, the stability of continuous–time singular systems is related to the uniqueness of the solutions of the Lyapunov equations, provided that the systems are controlable. Furthemore, under certain conditions, the controllability Grammians obtained from the Lyapunov equations are guaranteed to be positive definite. All the results are valid for both impulsive and non–impulsive singular systems. Many definitions of cotrollability of the infinite–frequency modes of singular systems have been presented in the literature. However, for time–invariant systems with a regular pencil (sE – A), all these definitions reduce down to two definitions of controllability at infinity. These are analogous to the difference between controllability and reachability.

The parameters of the Laurent expansion of the generalized resolvent matrix $(sE - A)^{-1}$ are a very useful tool for analyzing singular systems. This is because they separate the subspace spanned by solutions in the eigenspace associated with finite eigenvalues of the pencil (sE – A) from the subcpace spanned by solutions associated with infinite eigenvalues. The infinite–eigenspace solutions can be termed as a "impulsive" solutions in a continuous–time system.

The Laurent parameters can thus be used to split the system, given by Eq. (2) into causal (nonimpulsive) and noncausal (impulsive) subsystems.

The Laurent parameters, also known as fundamental matrices, have played an important part in the analysis of singular systems. Based on these parameters, Lewis (1985) defined the controllability matrices for the analysis of the controllability of descriptor systems. Bender (1987) introduced the reachability Grammians and associated them with Lyapunov–like equations without the non–impulsive or causality restriction.

Suppose that (sE – A) is a regular pencil. The system, given by Eq. (2) is denoted by (E, A, B, C). It is known that the Laurent parameters $\{\phi_k, -\mu \le k < \infty\}$ specify the unique series expansion of the resolvent matrix about $s = \infty$

$$(sE - A)^{-1} = s^{-1} \sum_{k=-\mu}^{\infty} \phi_k s^{-k}, \quad \mu \ge 0 \tag{73}$$

valid in some set $0 < |s| \le \delta$, $\delta > 0$. The positive integer μ is the nilpotent index. Two square invertible matrices U and V exist such that (E, A, B, C) is transformed to the Weierstrass canonical form:

$$\begin{aligned}\bar{E} &= U^{-1}EV^{-1}, \; \bar{A} = U^{-1}AV^{-1}\\ \bar{B} &= U^{-1}B, \; \bar{C} = CV^{-1}\end{aligned} \tag{74}$$

with:

$$\begin{aligned}&s\bar{E} - \bar{A} = \begin{bmatrix} sI-J & 0 \\ 0 & sN-I \end{bmatrix},\\ &\bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^T,\end{aligned} \tag{75}$$

where J and N are in the Jordan canonical form and N is nilpotent. Also, the corresponding Laurent parameters in Weierstrass form are:

$$\bar{\phi}_k = V\phi_k U = \begin{cases} \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix}, & k \ge 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix}, & k < 0 \end{cases} \tag{76}$$

Remark 1. If E is nonsingular, the singular system, given by Eq. (2) can be premultiplied by E^{-1} to derive an equivalent state–space system. In this case the following simplifications occur:

$$\begin{aligned}&\phi_o = I, \; U = E, \; V = I, \; J = E^{-1}A,\\ &B_1 = E^{-1}B, \; C_1 = C,\end{aligned} \tag{77}$$

and N, B_2 and C_2 do not exist (i.e., N is a zero–dimensional matrix). In this case the eigenvalues of the pencil (sE – A) are the eigenvalues of $E^{-1}A$ and are obviously finite. If E = I, Eq. (2) is already in the Weierstrass canoical form and one can have:

$$U = I, \; J = A, \text{ and } B_1 = B. \tag{78}$$

We now summarize some useful propreties of the Laurent parameters:

$$E\phi_k - A\phi_{k-1} = \phi_k E - \phi_{k-1}A = \delta_{ok} I \tag{79}$$

$$\phi_o E \phi_o = \phi_o \tag{80}$$

$$\phi_{-1} A \phi_{-1} = -\phi_{-1} \tag{81}$$

$$\phi_k = \begin{cases} (\phi_o A)^k \phi_o, & k \ge 0 \\ (-\phi_{-1}E)^{-k-1}\phi_{-1}, & k < 0 \end{cases} \tag{82}$$

$$E\phi_k A = A\phi_k E, \text{ for all } k \tag{83}$$

$$\phi_k E \phi_j = \phi_j E \phi_k = \phi_k A \phi_j = \phi_j A \phi_k$$

if:

$$k < 0, \quad j \ge 0 \tag{84}$$

$$\begin{aligned}&(-\phi^{-1}E)^{\mu} = (-E\phi^{-1})^{\mu} = 0\\ &(-\phi_{-1}E)^{\mu-1} \ne 0, \; (-E\phi_{-1})^{\mu-1} \ne 0\end{aligned} \tag{85}$$

$\phi_o E$ and $E\phi_o$ are projections on H_F along H_1
$-\phi_{-1}A$ and $A\phi_{-1}$ are projections on H_1 along H_F (86)

where H_F and H_1 the spaces spanned by the eigenvectors v_i satisfying $\lambda_i E v_i = A v_i$ corresponding to the finite and infinite eigenvalues λ_i, respectively. That is, H_F is the subspace spanned by causal solutions and H_I is the subspace spanned by noncausal or "infinite frequency" or "impulsive" solutions. Note that if E is nonsingular, $H_F = \mathbf{R}^n$, $H_I = 0$, $\phi_o = I$, $\phi_o E = E = E\phi_o$, and $\phi_{-1} = \phi_{-1}A = A\phi_{-1} = 0$.

The solution of a singular system can be expressed directly in terms of the Laurent parameters.

$$\mathbf{x} = \phi_o E\mathbf{x} - \phi_{-1} A\mathbf{x}(t) =$$
$$= \left(e^{\phi_o A t}\mathbf{x}_o + \int_0^t e^{\phi_o A(t-\tau)}\phi_o B\mathbf{u}(\tau)d\tau \right) -$$
$$\left((-\phi_{-1}E)^m \mathbf{x}^{(m)}(t) + \sum_{k=0}^{m-1} (-\phi_{-1}E)^k \phi_{-1}B\mathbf{u}^{(k)}(t) \right) \quad (87)$$

$$\mathbf{y}(t) = C(\phi_o E - \phi_{-1}A)\,\mathbf{x}(t) \quad (88)$$

where, $i \geq 0$ and $m \geq 0$. As indicated by the property of Eq. (87), the Laurent parameters can be used to separate the causal solution subspace from the noncausal solution subspace.

Definition 9. If the integral exists, the causal continuous–time singular system reachability Grammian is:

$$G_c^{cr} = \int_0^\infty \phi_o e^{A\phi_o t} BB^T e^{\phi_o^T A^T t}\phi_o^T\, dt\,. \quad (89)$$

Bender (1987).

The noncausal continuous–time singular system reachability Grammian is:

$$G_{nc}^{cr} = -\sum_{k=-mu}^{-1} \phi_k BB^T \phi_k^T\,. \quad (90)$$

The continuous–time singular system reachability Grammian is:

$$G^{cr} = G_c^{cr} + G_{nc}^{cr}\,. \quad (91)$$

If the integral does not exist, only G_{nc}^{cr} is defined, Bender (1987).

The columns of $\phi E G_c^c E^T \phi_o^T = G_c^{cr}$ span the causal reachable subspace, and the columns of G_{nc}^{cr} span the noncausal reachable subspace, which is the subspace "reachable at ∞". By the same argument the columns of G^{cr} span the reachable subspace for the entire system.

Theorem 14.

i) If G_c^{cr} exists, it satisfies
$$\phi_o (EG_c^{cr}A^T + AG_c^{cr}E^T)\,\phi_o^T = -\phi_o BB^T\phi_o^T\,. \quad (92)$$

ii) G_c^{cr} always exists and satisfies
$$\phi_{-1} (EG_c^{cr}E^T + AG_c^{cr}A^T)\,\phi_{-1}^T = -\phi_{-1}BB^T\phi_{-1}^T\,. \quad (93)$$

iii) Suppose the range of R^c (see Apendix B) contains the range of ϕ_E (i.e., the pair (J, B_1) is reachable). Then if all finite eigenvalues of the pencil (sE – A) have real part less than zero, eq. (92) has a symmetric solution G_c^{cr} which satisfies:

$\mathbf{x}^T G_c^{cr}\mathbf{x} > 0$ for all $\mathbf{x}$ such that:

$$\mathbf{x} = E^T\phi_o^T\,\mathbf{w} \neq \mathbf{0}. \quad (94)$$

Further, $\phi_o EG_c^{cr}E^T\phi_o^T$ is unique.

Conversely, if Eq. (92) has a symmetric solution satisfying eq. (94), then $\phi_o E\, G_c^{cr}\, E^T\phi_o^T$ is unique and all finite eigenvalues of the pencil (sE – A) have real part less than zero.

iv) If the rang of R_{nc} contains the range of $\phi_{-1}A$ (i.e., if the pair (N, B_2) is reachable), then Eq. (93) has a symmetric solution G_c^{cr} satisfying:

$\mathbf{x}^T G_{nc}^{cr}\,\mathbf{x} < 0$, for all $\mathbf{x}$ such that:

$$\mathbf{x} = A^T\phi_{-1}^T\,\ \mathbf{w} \neq \mathbf{0}. \quad (95)$$

Further, $\phi_{-1} AG_{nc}^{cr} A^T\phi_{-1}^T$ is unique.

For the sake of breavity the proof is ommited and can be found in Bender (1987).

Definition 10. A singular system is asymptotically stable if and only if its slow subsystem (I, J, B_1, C_1) is asymptotically stable. The slow subsystem is controllable, or equivalently, the singular system is R–controllable, if and only if:

$$\text{rank}\,[B_1, J B_1,..., J^{n}{}_1{}^1 B_1] = n_1 \quad (96)$$

where n_1 = degree(det(sE – A)) is the dimension of the slow subsystem. The fast subsystem is controllable if and only if:

$$\text{rank}\,[B_2, N B_2,..., N\mu^{-1} B_2] = n - n_1. \quad (97)$$

Dai (1989).

The controllability of a singular system implies both its slow and fast subsystems are controllable.

Definition 11. For the continuous–time descriptor system (E, A, B, C), the slow controllability Grammian is

$$G_s^c = \int_0^\infty \phi_o e^{A\phi_o t} BB^T e^{\phi_o^T A^T t}\phi_o^T\, dt\,, \quad (98)$$

provided that the integral exists. The fast controllability Grammian is:

$$G_f^c = \sum_{k=-\mu}^{-1} \phi_k BB^T\phi_k^T \quad (99)$$

The controllability Grammian is:

$$G^c = G_s^c + G_f^c\,, \quad (100)$$

Zhang et al. (1988b).

It can be seen that there is no significance difference between Definition 11 and Definition 9.

In Weierstrass canonical form, given by Eq. (75), the corresponding Grammians of G_s^c and G_f^c are denoted by $\bar{G}_s^c$ and $\bar{G}_f^c$ respectively. From Eq. (75) and Eq. (76), it can be easily shown that:

$$\bar{G}_s^c = VG_s^c V^T,\ \bar{G}_f^c = VG_f^c V^T . \tag{101}$$

Proposition 1.

$$\text{i) } \phi E G_s^c E^T \phi_o^T = G_s^c , \tag{102}$$

$$\text{ii) } \phi_{-1} A G_f^c A^T \phi_{-1}^T = G_f^c . \tag{103}$$

Proof.

i) From Eqs. (79–84), one can have:

$$\phi_o E G_s^c E^T \phi_o^T = \int_0^\infty \phi_o e^{A\phi_o t} BB^T e^{\phi_o^T A^T} \phi_o^T E^T \phi_o^T dt =$$

$$= \int_0^\infty \phi_o e^{A\phi_o t} BB^T e^{\phi_o^T A^T} \phi_o^T dt = G_s^c . \tag{104}$$

ii) From Eqs. (79–84), one can also have:

$$\phi_{-1} A G_f^c A^T \phi_{-1}^T = \sum_{k=-\mu}^{-1} \phi_{-1} A \phi_k BB^T \phi_k^T A^T \phi_{-1}^T =$$

$$= \sum_{k=-\mu}^{-1} \phi_k BB^T \phi_k^T = G_f^c . \tag{105}$$

In relation to the Grammians defined for (E, A, B, C), the corresponding Lyapunov equations will be stated.

Theorem 15.

i) G_s^c satisfies

$$G_s^c A^T \phi_o^T + \phi_o A G_s^c = -\phi_o BB^T \phi_o^T . \tag{106}$$

ii) G_f^c uniquely satisfies:

$$G_f^c A^T \phi_{-1}^T + \phi_o A G_f^c = -\phi_{-1} BB^T \phi_{-1}^T . \tag{107}$$

iii) If the system, given by Eq. (2), is asymptotically stable, then the slow subsystem is controllable if and only if Eq. (106) has the unique solution 0 which satisfies:

$$\text{rank}(G_s^c) = \text{degree}(\det(sE - A)). \tag{108}$$

iv) The fast subsystem is controllable if and only if:

$$\text{rank}(G_f^c) = n - \text{degree}(\det(sE - A)). \tag{109}$$

v) If the system, given by Eq. (2), is asymptotically stable, then system given by Eq. (2), is controllable if and only if:

$$G^c = G_s^c + G_f^c > 0 . \tag{110}$$

Proof.

i) and ii) can be easily established from, Bender (1987), with Eq. (102).

iii) When Eq. (2) is in Weierstrass canonical form, given by Eq. (75), such that:

$$\bar{G}_s^c = \begin{bmatrix} G_{11}^c & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix}, \tag{111}$$

then Eq. (106) reduces to:

$$\begin{bmatrix} G_{11}^c & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix} \begin{bmatrix} J^T & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} x$$

$$\begin{bmatrix} G_{11}^c & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix} = \begin{bmatrix} -B_1 B_1^T & 0 \\ 0 & 0 \end{bmatrix} \tag{112}$$

That is:

$$G_{11}^c J^T + J G_{11}^c = -B_1 B_1^T , \tag{113}$$

$$J G_{12} = 0. \tag{114}$$

Since Eq. (2) is asymptotically stable, then $G_{12} = 0$ and it is obvious that $G_{11}^c > 0$ is the unique solution of Eqs. (113–114) if and only if the slow subsystem is controlable. Condition, given by Eq. (109) ensures that $G_{22} = 0$, and hence:

$$\bar{G}_s^c = \begin{bmatrix} G_{11}^c & 0 \\ 0 & 0 \end{bmatrix} \tag{115}$$

is the unique solution of eq. (112).

iv) When Eq. (2) is in Weierstrass canonical form, given by Eq. (75) such that:

$$\bar{G}_f^c = \begin{bmatrix} G_{11} & G_{21} \\ G_{21}^T & G_{22}^c \end{bmatrix}, \tag{116}$$

then Eq. (108) reduces to:

$$\begin{bmatrix} G_{11} & G_{21} \\ G_{21}^T & G_{22}^c \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & NG_{22}^c N^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_2 B_2^T \end{bmatrix} \tag{117}$$

Hence $G_{11} = G_{21} = 0$. Notice that N is nilpotent and $G_{22}^c \geq 0$ is the unique solution of:

$$G_{22}^c - NG_{22}^c N^T = B_2 B_2^T . \tag{118}$$

The uniqueness of:

$$\bar{G}_f^c = \begin{bmatrix} 0 & 0 \\ 0 & G_{22}^c \end{bmatrix} \tag{119}$$

then follows. Furthemore, $G_{22}^c > 0$ if and only if the fast subsystem is controllable, and now $\bar{G}_f^c$ satisfies (Eq. 109).

v) From Eq. (115) and Eq. (119), follows:

$$\bar{G}^c = \bar{G}_s^c + \bar{G}_f^c = \begin{bmatrix} G_{11}^c & 0 \\ 0 & G_{22}^c \end{bmatrix}. \tag{120}$$

If the system, given by Eq. (2), is controllable, both the slow and fast subsystem are controllable. Hence if system, given by Eq. (2), is stable, then Eq. (2) is controllable if and only if $\bar{G}^c > 0$.

Remark 2. If E is nonsingular, then ϕ_o = I and ϕ_{-1} = 0. In this case, the controllability Grammian G^c becomes

$$G^c = \int_0^{\infty} e^{At}BB^Te^{A^Tt}dt\,. \tag{121}$$

It can be seen that G^c satisfies:

$$G^cA^T + AG^c = -BB^T \tag{122}$$

Therefore, normal systems and singular systems have unified Grammian form and Lyapunov equations, Zhang et al (1988b).

CONCLUSSION

Singular systems are also present in processes and chemical engineering, see Bogdanović (1992), Lapidus et al. (1961) and Daley, Wang (1994). Some of the mathematical model have been shown to illustrate this fact.

To assure asymptotical stability for linear singular systems it is not enough only to have the eigenvalues of matrix pair (E,A) in the left half complex plane, but also to provide an impulse–free motion of the system under consideration. Some different approaches have been shown in order to construct Lyapunov stability theory for a particular class of linear singular systems operating in free and forced regimes.

APPENDIX A – Unussual notations

With $\aleph(F)$ and $\Re(F)$ we will denote the kernel (null space) and range on any operator F, respectively, i.e.:

$$\aleph(F) = \{\mathbf{x}: F\mathbf{x} = \mathbf{0},\ \forall\ \mathbf{x} \in \mathbf{R}^n, \tag{A1}$$

$$\Re(F) = \{\mathbf{y} \in \mathbf{R}^m,\ \mathbf{y} = F\mathbf{x},\ \mathbf{x} \in \mathbf{R}^n\}, \tag{A2}$$

with:

$$\dim \aleph(F) + \dim \Re(F) = n. \tag{A3}$$

APPENDIX B – Reachability Grammians

We begin this section by defining the reachable subspace in terms of the Laurent parameters. We follow the deveopment of Lewis (1985). We shall define the reachable subspace in terms of the following reachability matrices:

$$R_c = (\phi_o B \ \ldots\ \phi_{n-1}B), \tag{B1}$$

$$R_{nc} = (\phi_{-\mu}B \ \ldots\ \phi_{-1}B), \tag{B2}$$

and:

$$R = (R_{nc}\ \ R_c). \tag{B3}$$

The subscript c implies that the columns of R_c span the reachable part of the causal solution subspace, and the subscript nc implies that the columns of R_{nc} span the reachable part of the noncausal solution subspace.

Definition B1. For a continuous–time singular system, the causal reachable subspace is the space spanned by the columns of R_c, the noncausal reachable subspace is the space spanned by the columns of R_{nc}, and the reachable subspace is the space spanned by the columns of R, Lewis (1985).

Remark B1:

1) If the reachable subspace defined here for the continuus–time system, given by Eq. (2) is equal to $\mathbf{R}^n$, the singular system is "controllable" in the sense of Cobb (1984). That is a $(\mu - 1)$ – times continuously differentiable input $\mathbf{u}(t)$ exist which will steer the descriptor vector $\mathbf{x}(t)$ from any initial condition in the range of $\phi_o E$ to any arbitrary location in the descriptor space $\mathbf{R}^n$ in finite time. This is an extension of (and if E = I is equivalent to) the usual definition of reachability for state–space systems.

2) If and only if the causal subsystem is reachable, i.e., if the pair (J, B_1) is reachable, do the columns of span R_c the range of $\phi_o E$. That is, the columns of R_c span the causal solution subspace.

3) If and only if the noncausal subsystem is reachable, i.e., if the pair (N, B_2) is reachable, do the columns of R_{nc} span the range of $\phi_{-1}A$. That is, the columns of R_{nc} span the noncausal solution subspace.

REFERENCES

[1] Bajić, V.B., Lyapunov's Direct Method in The Analysis of Singular Systems and Networks, Shades Technical Publications, Hillcrest, Natal, RSA, 1992.

[2] Bajić, V.B., D. Debeljković, Z. Gajić, B. Petrović, "Weak Domain of Attraction and Existence of Solutions Convergent to the Origin of the Phase Space of Singular Linear Systems", University of Belgrade, ETF, Series: Automatic Control, (1) (1992.b) 53–62.

[3] Bender, D.J., "Lyapunov – Like Equations and Reachability / Observability Gramians for Descriptor Systems", IEEE Trans. Automat. Cont., AC–32 (4) (1987) 343–348.

[4] Bogdanović, S., "Nonlinear Mathematical Model of Steam Boiler Superheating Column", Proc. Inst. Nikola Tesla, Belgrade, (1992) 253 – 272

[5] Campbell, S.L., Singular Systems of Differential Equations, Pitman, Marshfield, Mass., 1980.

[6] Campbell, S.L., Singular Systems of Differential Equations II, Pitman, Marshfield, Mass., 1982.

[7] Chen, C., Y. Liu, "Lyapunov Stability Analysis of Linear Singular Dynamical Systems", Proc. Int. Conference on Inteliligent Processing Systems, Beijing, (China), October 28 – 31, (1997) 635–639.

[8] Circuits, Systems and Signal Processing, Special Issue on Semistate Systems, **5** (1) (1986).

[9] Circuits, Systems and Signal Processing, Special Issue: Recent Advances in Singular Systems, **8** (3) (1989).

[10] Cobb, D., "Controllability, Observability and Duality in Singular Systems", IEEE Trans. Automat. Cont., **AC–29** (12) (1984) 1076–1082.

[11] Dai, L., Singular Control Systems, Springer Verlag, Berlin, 1989.

[12] Daley, S., H. Wang, Proc. of ImechE – No. 207, (1993) 153–164.

[13] Daley, S., H. Wang, "Adaptive Gas Turbine Control using a Singular System Approach", Proc. of Contol 94, Conference Publication No. 389, 21–24 March (1994) 687 – 691.

[14] Debeljković, D., Lj. "Stabilty of Linear Continuous Singular Systems in the sense of Lyapunov: An Owerview ", Proc. SANU (YU), April 4 – 5, (2001).

[15] Debeljković, D.Lj., S.A. Milinković, M.B. Jovanović, Application of Singular Systems Theory in Chemical Engineering, MAPRET Lecture – Monograph, 12th International Congress of Chemical and Process Engineering, CHISA 96, Praha, Czech Republic 1996.

[16] Debeljković, Lj.D., S.A. Milinković, M.B. Jovanović, Continuous Singular Control Systems, GIP Kultura, Beograd, 1996.a.

[17] Debeljković, Lj.D., S.A. Milinković, M.B. Jovanović, Lj.A. Jacić, Discrete Singular Control System, GIP Kultura, Beograd, 1998.

[18] Foss, A.M., "Derivation of Transfer Function Model for a Gas Turbine", Memorandum 80020, National Gas Turbine Establishment (1980)

[19] Geerts, T., "Stabilty Concepts for General Continuous–times Implicit Systems: Definitions, Hautus Test and Lyapunov Criteria", Int. J. Systems Sci., 26 (3) (1995) 481–498.

[20] Lapidus, L., "Optimization of Process Performance", A.I.Ch.E.I. **7** (1961) 288 – 294.

[21] Lazarević, P.M., Debeljković, Lj.D., M.B. Jovanović, M.V. Rančić, V.S. Mulić, "Optimal Control of Linear Singular Systems with Pure Time Lag", NTP, (YU) (2001), (in press).

[22] Lewis, F.L., "Fundamental, Reachabilty and Observabilty Matrices for Descriptor Systems", IEEE Trans. Automat. Cont., **AC–30** (45) (1985) 502–505.

[23] Lewis, F.L., "A Survey of Linear Singular Systems", Circ. Syst. Sig. Proc., **5** (1) (1986) 3–36.

[24] Lewis, F.L., "Recent Work in Singular Systems", Proc. Int. Symp. on Sing. Syst., Atlanta, GA (1987) 20–24.

[25] Muller, P.C., "Stability of Linear Mechanical Systems with Holonomic Constraints", Appl. Mech. Rev. (11), part 2, November (1983), 160–164.

[26] Owens, D.H., D.Lj. Debeljković, "Consistency and Liapunov Stability of Linear Decsriptor Systems: a Geometric Approach", IMA Journal of Math. Control and Information, (1985), No. 2, 139 – 151.

[27] Pandolfi, L., "Controllability and Stabilization for Linear Systems of Algebraic and Differential Equations", JOTA, **30** (4) (1980) 601–620.

[28] Takaba, K., N. Morihira, T. Katayama, "A Generalized Lyapunov Theorem for Descriptor System", Systems &Control Letters, (24) (1995) 49–51.

[29] Verghese, G.C., B.C. Levy, T. Kailath, "A Generalized State–Space for Singular Systems", IEEE Trans. Automat. Cont., **AC–26** (4) (1981) 811–831.

[30] Wang, H., S. Daley, Int. J. Systems Sci., No. 24, (1993) 1791–1801.

[31] Wu, H., K. Muzukami, "Lyapunov Stability Theory and Robust Control of Uncertain Descriptor Systems", Int. J. Science, Vol. 26, (10), (1995), 1981–1991.

[32] Zhang, Q., G. Dai, J. Lam, L.Q. Zhang, M. De La Sen, "Asymptotical Stability and Stabilization of Descriptor Systems", Acta Automatica Sinica, Vol. 24 (2), (1998a), 208–211.

[33] Zhang, L., J. Lam, Q. Zhang, "New Lyapunov and Riccati Equations for Descriptor Systems: Continuous – Time Case ", Proc. Fifth ICARCV 98, Singapore, December (1998b) TP1.5 – 965–969.

IZVOD

TEORIJA SINGULARNIH SISTEMA U HEMIJSKOM INŽENJERSTVU – PREGLED U ODNOSU NA STABILNOST PO LJAPUNOVU

(Pregledni rad)

Dragutin Lj. Debeljković[1], Mića B. Jovanović[2], Vesna Drakulić[1]
[1]Mašinski fakultet, Beograd, [2]Tehnološko–metalurški fakultet, Beograd, Jugoslavija

Singularni sistemi predstavljeni su u matematičkom smislu kombinacijom diferencijalnih i algebarskih jednačina, pri čemu ove druge predstavljaju ograničenje koje diferencijalni deo i njegovo rešenje mora da zadovolji u svakom trenutku. Singularni sistemi prisutni su u svim granama nauke i tehnike.

U ovom radu navedeni su brojni primeri singularnih sistema koji se susreću u hemijskoj i procesnoj industriji. Pored toga dat je iscrpan hronološki pregled postojećih rezultata na polju ispitivanja stabilnosti ove klase sistema sa pozicija Ljapunova što sigurno predstavlja nezaobilazni korak u dinamičkom ispitivanju svakog sistema automatskog upravljanja.

Ključne reči: Singularni sistemi • Matematičko modeliranje • Ljapunovska stabilnost • Ljapunovljeva matrična jednačina.

Key words: Singular systems • Mathematical modelling • Lyapunov stability • Lyapunov matrix equation •